Lattices of continuous and uniformly continuous functions

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On characterizing innerly the set $C(X)$ of real-valued continuous functions on a topological space $X$ depends essentially on the algebraic structure we are interested in.

Whenever $X \neq \emptyset$, the set $C(X)$ endowed with its pointwise defined order becomes a distributive lattice containing all the constant functions into $\mathbb{R}$, thus a copy of $\mathbb{R}$ as a sublattice. In the sequel, our basic structure on $C(X)$ will be that of “real lattice”. Without losing generality we may assume $X$ is completely regular, Hausdorff and even realcompact (since $C(X)$ is lattice-ordered algebra unit preserving isomorphic to $C(\mu X)$). At the crux of most attempts, if $L^*$ denotes the bounded elements of a given real lattice $L$, then the following conditions somehow are needed: (a) $L$ embeds into some $C(X)$, and (b) $L^*$ is isomorphic to $C^*(X)$.

Under this general context, the unique contribution appearing in the literature for the more general case is that of Jensen [9] as a refinement that of Anderson [2], but by assuming richer compatible algebraic structures, namely for $\Phi$-algebras. Our effort to weaken algebraic assumptions lead to characterizations of $C(X)$ as Riesz spaces [10, 8], real $\ell$-groups [3] or semi-affine lattices [4]. However, the nicest and purely lattice-theoretic real lattices characterization of $C(X)$ for $X$ a compact Hausdorff space is that of Anderson-Blair [1]. From this seminal result, and by adding convenient conditions we may characterize $C(X)$ as a real lattice [5] by constituting a generalization of the famous problem 81 of Birkhoff and Kaplansky (1948). In [6] we present a unified approach valid for any “convenient” subcategory of the real lattices.

By identifying the Samuel’s compactification of a uniform space by means of certain partitions of a real lattice, and by setting equivalent conditions to equi-uniform continuity, a characterization of the real lattice $U(X)$ of real-valued uniformly continuous functions on a uniform space $X$ is derived from [7].

References


